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Virtual turning points and bifurcation of Stokes curves for higher order ordinary differential equations

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Abstract
For a higher order linear ordinary differential operator P, its Stokes curve bifurcates in general when it hits another turning point of P. This phenomenon is most neatly understandable by taking into account Stokes curves emanating from virtual turning points, together with those from ordinary turning points. This understanding of the bifurcation of a Stokes curve plays an important role in resolving a paradox recently found in a system of linear differential equations associated with the fourth Painlevé equation.

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Introduction

The aim of this paper is to report an illuminating example encountered in the study of Painlevé equations, which manifests the importance of virtual turning points [1] in WKB analysis of higher order ordinary differential equations. The notion of virtual turning points, to be recalled in definition 1.1, is closely tied up with the exact WKB analysis, that is, WKB analysis based on the Borel resummation. Exact WKB analysis has been effectively applied to many problems in mathematical physics [2], and we hope this paper may be counted as another example that shows the importance of exact WKB analysis in mathematical physics. The advantage in employing the Borel resummation method in WKB analysis certainly consists of its efficiency in manipulating exponentially small terms, but still more important, from the theoretical viewpoint, is the fact that the Borel transform $P_B\left(x, \partial_y^{-1}\partial_x\right)$ of an ordinary differential operator $P(x, \eta^{-1}\,\mathrm{d}/\mathrm{d}x)$ with a large parameter η is a partial differential (or microdifferential) operator on (x, y)-space with y denoting the variable dual to η. Fortunately, a new and powerful machinery called microlocal analysis [3] is available to analyse linear partial differential (or, more generally, microdifferential) operators; it is particularly efficient in analysing the singularity structure of solutions of the equation $P_B\varphi = 0$, and the notion of virtual turning points originates from such analysis [1, 4, 5]. As we will see later in definition 1.1, the most important keyword in our analysis is a bicharacteristic curve associated with the Borel transform P_B of the operator P in question; microlocal analysis guarantees that it is the most 'elementary' carrier of singularities of solutions of linear partial differential equations in general [3]. Note that Voros [6] uses the corresponding result for the Tricomi-type operator in constructing his theory of exact WKB analysis for differential operators of the second order. As the so-called new Stokes curve for a higher order operator [7] is nothing but an ordinary Stokes curve emanating from a virtual turning point, the importance of the notion of a virtual turning point is practically evident. Actually it plays an important role in computing the transition probabilities for the non-adiabatic transition problems of the Landau–Zener type [4]. Here in this paper we present a decisive and illuminating evidence of the theoretical importance of the notion in a concrete example encountered in the study of the fourth Painlevé equation written in a symmetric form [8]. In order to facilitate the reader's understanding of the implication of the example, we first recall some basic facts in the exact WKB analysis of the Painlevé equations which are immediately related to the example. (Although we discuss below equations related to the fourth Painlevé equation, we can discuss all $(P_J)(J = \mathrm{I}, \mathrm{II}, \ldots, \mathrm{VI})$ with appropriate modifications. See [9] for details.)

Let us start with the following Schrödinger equation (SL_{IV}):

$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\mathrm{IV}}(x,t,\eta)\right)\psi(x,t,\eta) = 0, \tag{SL_{IV}}$$

where

$$Q_{\mathrm{IV}} = \frac{\alpha}{x^2} + \beta + \left(\frac{x+2t}{4}\right)^2 + \frac{K_{\mathrm{IV}}}{2x} - \eta^{-1}\frac{\lambda\nu}{x(x-\lambda)} + \eta^{-2}\frac{3}{4(x-\lambda)^2}, \tag{0.1}$$

with α, β constants and the Hamiltonian K_{IV} given by

$$2\lambda\left[\nu^2 - \eta^{-1}\frac{\nu}{\lambda} - \left(\frac{\alpha}{\lambda^2} + \beta + \left(\frac{x+2t}{4}\right)^2\right)\right]. \tag{0.2}$$

Here λ and ν are functions of t and η.

We next consider its deformation equation (D_{IV}):

$$\frac{\partial\psi}{\partial t} = A\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial A}{\partial x}\psi, \tag{D_{IV}}$$

where

$$A = \frac{2x}{x-\lambda}. \tag{0.3}$$

Then the compatibility condition of (SL_{IV}) and (D_{IV}) can be written in a Hamiltonian form [10] as follows:

$$\begin{cases} \dfrac{\mathrm{d}\lambda}{\mathrm{d}t} = \eta\dfrac{\partial K_{\mathrm{IV}}}{\partial\nu} \\ \dfrac{\mathrm{d}\nu}{\mathrm{d}t} = -\eta\dfrac{\partial K_{\mathrm{IV}}}{\partial\lambda}. \end{cases} \tag{0.4}$$

The fourth Painlevé equation (P_{IV}) given in (0.5) then follows from (0.4):

$$\frac{\mathrm{d}^2\lambda}{\mathrm{d}t^2} = \frac{1}{2\lambda}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right)^2 - \frac{2}{\lambda} + \eta^2\left[\frac{3}{2}\lambda^3 + 4t\lambda^2 + (2t^2 + 8\beta)\lambda - \frac{8\alpha}{\lambda}\right]. \tag{0.5}$$

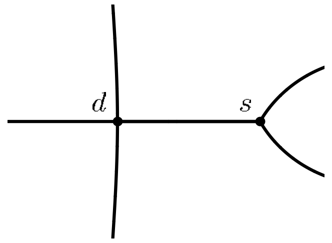

Figure 1. The degeneracy of the Stokes geometry of $(SL_{\rm IV})$ when the parameter t lies in the Stokes curve of $(P_{\rm IV})$ (with the choice of parameters $(t, \alpha, \beta) = (4, 1, 1)$).

Substituting into the coefficients of $(SL_{\rm IV})$ a particular solution (λ, ν) of (0.4) that has the form

$$\lambda = \lambda_0(t) + \eta^{-1}\lambda_1(t) + \eta^{-2}\lambda_2(t) + \cdots \tag{0.6}$$

$$\nu = \nu_0(t) + \eta^{-1}\nu_1(t) + \eta^{-2}\nu_2(t) + \cdots, \tag{0.7}$$

we obtain the expansion of $Q_{\rm IV}$ in η^{-1} and we let $Q_0 = Q_{{\rm IV},0}$ denote its top degree (i.e., degree 0) part. We can then consider the Stokes geometry of $(SL_{\rm IV})$ by using Q_0. The Stokes geometry, i.e., the collection of turning points and Stokes curves, is then drawn in the x-space, and it depends on the parameter t. Now, when the parameter t lies on a Stokes curve for $(P_{\rm IV})$, we observe an interesting degeneracy of the Stokes geometry of $(SL_{\rm IV})$ in the following sense:

A double turning point $d = d(t)$ of $(SL_{\rm IV})$ and a simple turning point $s = s(t)$ of $(SL_{\rm IV})$ are connected by a Stokes segment of $(SL_{\rm IV})$. (Cf figure 1.) (0.8)

A critically important role is played by the two turning points d and s which are relevant to this degeneracy of the Stokes geometry of $(SL_{\rm IV})$ in the exact WKB analysis of the Painlevé transcendents [9]. In particular, the integral

$$\phi(t) = \int_{d(t)}^{s(t)} \sqrt{Q_0(x, t)}\,{\rm d}x \tag{0.9}$$

is an essential ingredient in describing instanton-type solutions of (0.4), which are used to describe the connection formula for the Painlevé transcendents [9, 11]. We also note that the Stokes curve for $(P_{\rm IV})$ is given by

$$\operatorname{Im}\phi(t) = 0. \tag{0.10}$$

See [9] for details.

Now, if we study the fourth Painlevé equation written in a symmetric form (cf $(NY)_2$ in section 2) with the pair of 'Schrödinger' equation (2.3) and its deformation equation (2.4) given by Noumi and Yamada [12], we encounter the following intriguing situation [13]:

Fact A (found with the help of a computer). *When the parameter t lies in some portion* σ_1 *of a Stokes curve for the fourth Painlevé equation* $(NY)_2$*, there is no pair of turning points of the 'Schrödinger' equation* (2.3) *that are connected by a Stokes segment of* (2.3)*. At the same time, if the parameter t lies in another portion* σ_0 *of the same Stokes curve for the Painlevé equation, we observe a pair of turning points d and* s_1 *of* (2.3) *which are connected by a Stokes segment of* (2.3).

The situation is visualized by figure 2 concretely. We have included in the figure another nearby turning point s_2 for the later reference. The figure is drawn with the choice of parameters $(\alpha_0, \alpha_1) = (1.0 + 0.6\sqrt{-1}, 0.2 - 0.1\sqrt{-1})$.

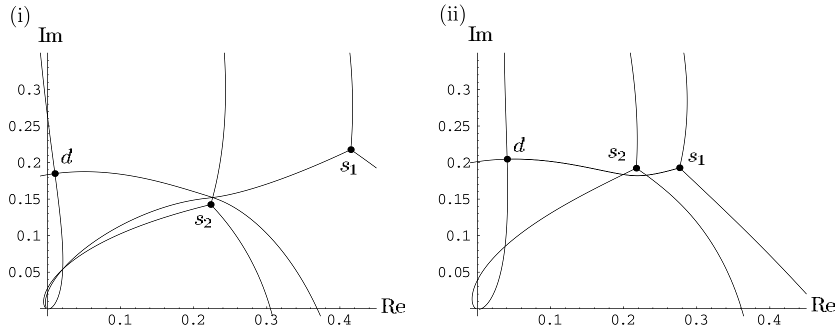

Figure 2. Configuration of Stokes curves of (2.3) for (i) $t = t_1 = -1.5783 - 0.4130\sqrt{-1}$ in σ_1 and (ii) $t = t_0 = -1.6104 - 0.2268\sqrt{-1}$ in σ_0.

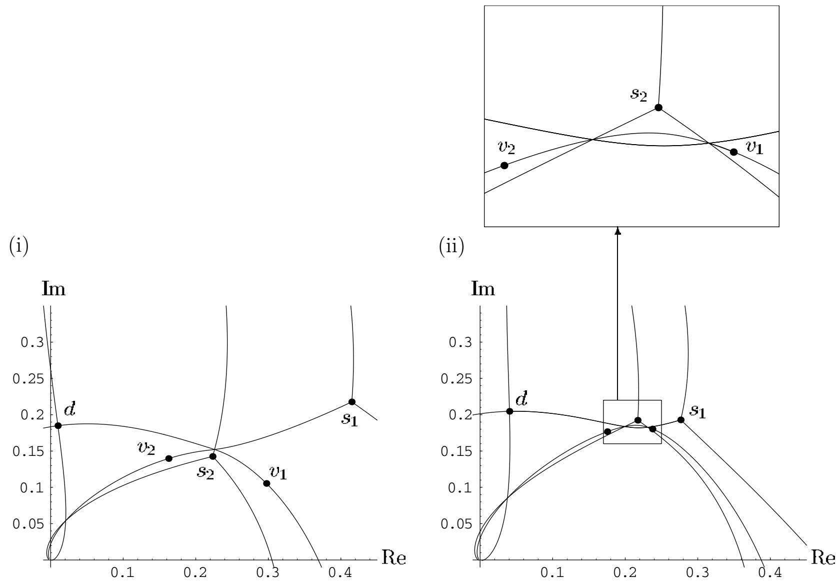

Figure 3. Stokes geometry of (2.3) with virtual turning points v_1 and v_2 added. Figure (i) is for $t = t_1$ and figure (ii) is for $t = t_0$.

This seemingly paradoxical situation might make the reader suspect that the 'Schrödinger' equation (2.3), which is of size 3×3, not 2×2, is not suited for WKB analysis. Indeed, the suspicion is reasonable if we use ordinary WKB analysis. However, we are armed with the exact WKB analysis; in particular, we have virtual turning points in our toolbox. If we add virtual turning points in figure 2, marvellous harmony is recovered:

Fact B (found with the help of a computer in [13]). *The double turning point d (resp., simple turning point s_1) is connected with a virtual turning point v_1 (resp., v_2) at $t = t_1$.*

This fact B is visualized by figure 3.

The comparison of fact A and fact B is more than enough to convince the reader that the notorious difficulty in the global asymptotic analysis of higher order differential equations (e.g. [14 p 176]) originates in the lack of the notion of virtual turning points in the traditional WKB analysis.

In order to understand the mechanism that switches the counterpart of the turning point d (resp., s_1) from an ordinary turning point s_1 (resp., d) to a virtual turning point v_1 (resp., v_2), we study in section 1 the relevance of a virtual turning point to the bifurcation phenomenon of a Stokes curve that is observed when it hits a simple turning point. In section 2, we discuss concretely how the mechanism analysed in section 1 is related to the change of configuration of turning points and Stokes curves of (2.3), which is displayed in figure 3. In appendix A we prove that the apparently miraculous situation observed in figure 8, i.e., one Stokes curve passing through two crossing points of Stokes curves simultaneously, is actually a phenomenon that occurs generically in our context. In appendix B we show an analytic counterpart of the geometric result discussed in section 2. Logically speaking, the contents of appendices A and B are not needed for the understanding of the main text of this paper. We hope, however, they will give the reader some insight into the analytic background of the subject discussed in this paper.

1. The relevance of a virtual turning point to the bifurcation phenomenon of a Stokes curve

To fix the notation and terminologies, we first recall the definition of a turning point, both ordinary and virtual, and a Stokes curve for an mth ($m \geqslant 3$) order ordinary differential operator $P = P(x, \eta^{-1}\,\mathrm{d}/\mathrm{d}x, \eta^{-1})$ with a large parameter η that is of the form

$$P_0(x, \eta^{-1}\,\mathrm{d}/\mathrm{d}x) + \eta^{-1}P_1(x, \eta^{-1}\,\mathrm{d}/\mathrm{d}x) + \eta^{-2}P_2(x, \eta^{-1}\,\mathrm{d}/\mathrm{d}x) + \cdots. \tag{1.1}$$

For such an operator P, a solution $\zeta(x)$ of the equation

$$P_0(x, \zeta) = 0 \tag{1.2}$$

is called a characteristic root of P. When we deal with a matrix equation of the form

$$\left(\eta^{-1}\frac{\mathrm{d}}{\mathrm{d}x} - A(x, \eta)\right)\psi = 0, \tag{1.3}$$

where A is an $m \times m$ matrix ($m \geqslant 3$) of the form

$$A_0(x) + \eta^{-1}A_1(x) + \eta^{-2}A_2(x) + \cdots, \tag{1.4}$$

we use $\det(\zeta - A_0(x))$ as $P_0(x, \zeta)$. In defining a virtual turning point we make use of a classical notion (e.g. [15 p 558 ff]) of bicharacteristic strip of the Borel transform $P_B\big(x, \partial_y^{-1}\partial_x, \partial_y^{-1}\big)$ of $P(x, \eta^{-1}\,\mathrm{d}/\mathrm{d}x, \eta^{-1})$. In what follows we identify η with the symbol of the operator ∂_y [3]. A bicharacteristic strip is, by definition, a solution $(x(s), y(s), \xi(s), \eta(s))$ of the following Hamilton–Jacobi equation determined in terms of the principal symbol of P_B, that is, $P_0 = P_0(x, \eta^{-1}\xi)$:

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}s} = \dfrac{\partial P_0}{\partial \xi} \\ \dfrac{\mathrm{d}y}{\mathrm{d}s} = \dfrac{\partial P_0}{\partial \eta} \\ \dfrac{\mathrm{d}\xi}{\mathrm{d}s} = -\dfrac{\partial P_0}{\partial x} \\ \dfrac{\mathrm{d}\eta}{\mathrm{d}s} = -\dfrac{\partial P_0}{\partial y}(= 0) \\ P_0(x(s), \eta(s)^{-1}\xi(s)) = 0, \end{cases} \tag{1.5}$$

and it defines a curve in the cotangent bundle $T^*\mathbb{C}^2_{(x,y)}$. The projection of a bicharacteristic strip to the base manifold $\mathbb{C}^2_{(x,y)}$ is called a bicharacteristic curve. As is seen in definition 1.1,

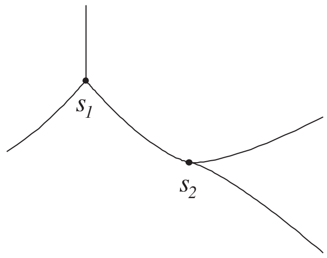

Figure 4. Bifurcation of a Stokes curve.

the notion of a bicharacteristic curve is essentially used in the definition of a virtual turning point.

Definition 1.1. *(i) When two characteristic roots $\zeta_j(x)$ and $\zeta_k(x)$ coalesce at $x = a$, the point a is called an ordinary turning point. If necessary, we put a sign (j, k) to a to indicate that it is determined by a pair of characteristic roots ζ_j and ζ_k. According as the discriminant of P_0 has a simple or double zero at $x = a$, a is called a simple or double ordinary turning point. If there is no fear of confusions the adjective 'ordinary' is omitted.*

(ii) When a bicharacteristic curve of P crosses itself at a point (x_0, y_0), the point x_0 is called a virtual turning point [1, 5]. When the crossing point is determined by a pair of Hamiltonians $(\eta^{-1}\xi - \zeta_j(x))$ and $(\eta^{-1}\xi - \zeta_k(x))$, we put a sign (j, k) to the virtual turning point.

(iii) An integral curve of the direction field

$$\mathrm{Im}(\zeta_j(x) - \zeta_k(x))\,\mathrm{d}x = 0 \tag{1.6}$$

that emanates from a turning point a, either ordinary or virtual, with the sign (j, k) is called a Stokes curve of type (j, k). The portion of a Stokes curve where

$$\mathrm{Re}\int_a^x (\zeta_j - \zeta_k)\,\mathrm{d}x > 0 \tag{1.7}$$

holds is labelled as $(j > k)$, or simply $j > k$. A Stokes curve that emanates from a virtual turning point is called a new Stokes curve, if necessary.

In what follows we freely use these terminologies and notation. Let us now consider the situation described in figure 4, that is, the situation where a Stokes curve emanating from a turning point s_1 with a sign $(j, k) = (1, 2)$ hits another simple ordinary turning point s_2 with a sign $(2, 3)$. Here, using the assumption $m \geqslant 3$, we have chosen mutually distinct characteristic roots $\zeta_j(x) (j = 1, 2, 3)$ of the operator P in question. Since $x = s_2$ is a simple turning point where $\zeta_2(x)$ and $\zeta_3(x)$ coalesce, $\zeta_2(x)$ (and $\zeta_3(x)$ also) has a square-root-type singularity there. As the Stokes curve emanating from s_1 is a solution curve of the direction field

$$\mathrm{Im}(\zeta_1(x) - \zeta_2(x))\,\mathrm{d}x = 0, \tag{1.8}$$

it bifurcates at $x = s_2$, forming the contact of degree $3/2$.

If the operator P does not contain any parameter other than η, one might be content to regard this bifurcation just as one of the pathologies which analysis of higher order equation presents. Then the reasoning would be stopped there. However, if the operator P depends on an auxiliary parameter t, it is natural to consider how the configuration of Stokes curves changes as the parameter t changes. Then it is more reasonable to take into account the Stokes

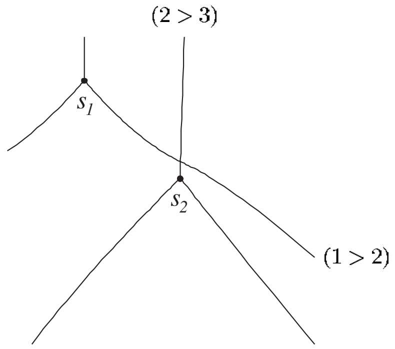

Figure 5. Configuration of Stokes curves for $t = t_2$.

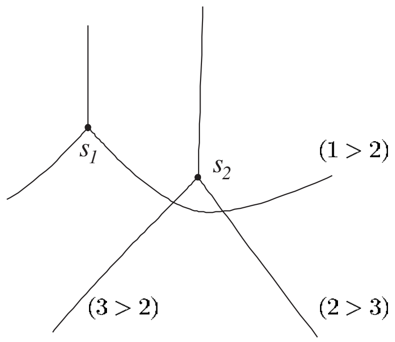

Figure 6. Configuration of Stokes curves for $t = t_3$.

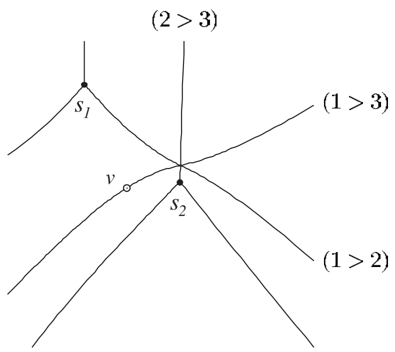

Figure 7. Figure 5 with a virtual turning point added.

curve emanating from $x = s_2(t)$, in addition to the Stokes curve emanating from $s_1(t)$. To fix the situation, let us suppose that $s_1(t)$ is a simple ordinary turning point. For the sake of definiteness we assign the dominance relation symbols $(1 > 2)$ etc as in figures 5 and 6, which respectively describe the configuration of Stokes curves at $t = t_2$ and $t = t_3$. Here the points $t = t_2$ and $t = t_3$ are chosen sufficiently close to $t = t_1$, at which the configuration of Stokes curves is that given in figure 4, that is, the Stokes curve emanating from $s_1(t_1)$ hits the simple ordinary turning point $s_2(t_1)$. In the situation described in figure 5, we can locate, as shown in figure 7, a virtual turning point $v(t_2)$ with sign $(1, 3)$ such that the Stokes curve emanating

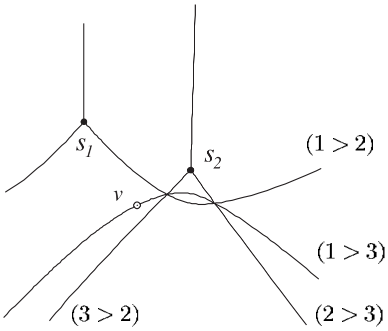

Figure 8. Figure 6 with the virtual turning point $v(t_3)$ added.

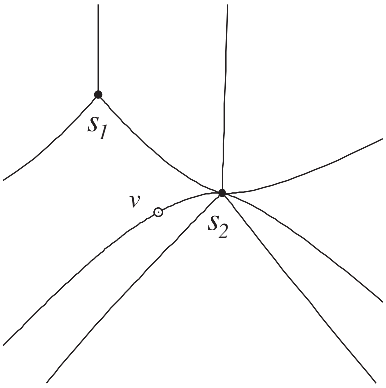

Figure 9. Figure 4 with the virtual turning point $v(t_1)$ added.

from $v(t_2)$ passes through the crossing point of the Stokes curve emanating from $s_1(t_2)$ and that from $s_2(t_2)$ [5, 13, 16].

The configuration of the Stokes curves at $t = t_3$ then becomes as in figure 8.

Since the Stokes curve emanating from $v(t_1)$ is with sign $(1, 3)$, it also bifurcates at $x = s_2(t_1)$ because of the square-root-type singularity that $\zeta_3(x)$ contains. The resulting configuration is given in figure 9.

Remark 1.1. It is worth emphasizing that the pattern of the change of configuration of Stokes curves observed above is a universal one in our context, if t_2 and t_3 are sufficiently close to t_1. See appendix A for details.

Comparison of the pair O of figures 5 and 6 and the pair A of figures 7 and 8 is instructive: in the pair O where only ordinary turning points are taken into account, the direction of the Stokes curve emanating from $x = s_1(t)$ abruptly changes as t crosses $t = t_1$, while, in the pair A where a virtual turning point and a Stokes curve emanating from it are added, the totality of directions of Stokes curves changes smoothly near $t = t_1$. The interest of this comparison is enhanced if one notes the fact that the relative location of the Stokes curve emanating from $s_1(t)$ and that from $v(t)$ are interchanged on the right of their crossing points. Thus we understand that the bifurcation of a Stokes curve is naturally coupled with the addition of a Stokes curve emanating from a virtual turning point; it is not an isolated pathology.

2. A pathology related to the Noumi–Yamada system and its resolution by the introduction of virtual turning points

To begin with, let us recall the explicit form of the Noumi–Yamada system $(NY)_2$ and its underlying Lax pair of 'Schrödinger' equation (2.3) and its deformation equation (2.4).

$$\eta^{-1}\frac{\mathrm{d}f_j}{\mathrm{d}t} = f_j(f_{j+1} - f_{j+2}) + \alpha_j \qquad (j = 0, 1, 2), \tag{$(NY)_2$}$$

where

$$f_j = f_{j-3} \quad (j = 3, 4) \tag{2.1}$$

and α_j $(j = 0, 1, 2)$ are constants satisfying

$$\alpha_0 + \alpha_1 + \alpha_2 = \eta^{-1}, \tag{2.2}$$

$$-\eta^{-1}x\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}(2\alpha_1+\alpha_2)/3 & f_1 & 1\\ x & (-\alpha_1+\alpha_2)/3 & f_2\\ xf_0 & x & -(\alpha_1+2\alpha_2)/3\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}, \tag{2.3}$$

$$\eta^{-1}\frac{\partial}{\partial t}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}f_2 - t/2 & -1 & 0\\ 0 & f_0 - t/2 & -1\\ -x & 0 & f_1 - t/2\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\\ \psi_2\end{pmatrix}. \tag{2.4}$$

Here a Lax pair means, in general, a pair of linear differential equations whose compatibility condition is the starting nonlinear equation, i.e., $(NY)_2$ in our case.

Remark 2.1. The Noumi–Yamada system $(NY)_2$ discussed in this paper is the first member of their hierarchy $\{(NY)_l\}_{l=2,3,4,\ldots}$ [12]. It gives a higher order version of the fourth Painlevé equation (for l even) and the fifth Painlevé equation (for l odd). The system $(NY)_2$ coincides with the fourth Painlevé equation written in a symmetric form [8]. One can readily verify that a component, say the first component, of the solution of $(NY)_2$ satisfies

$$f_1'' = \frac{1}{2f_1}(f_1')^2 + \eta^2\left(\frac{3}{2}f_1^3 - 2tf_1^2 + \left(\frac{t^2}{2} - \alpha_0 + \alpha_2\right)f_1 - \frac{\alpha_1^2}{2f_1}\right), \tag{2.5}$$

which reduces to the fourth Painlevé equation (0.5) by the scaling: $t \mapsto \sqrt{2}t$, $f_1 \mapsto -\lambda/\sqrt{2}$, and the choice of parameters

$$\alpha = \left(\alpha_1^2 - \eta^{-2}\right)/4, \tag{2.6}$$

$$\beta = (-\alpha_0 + \alpha_2)/4. \tag{2.7}$$

As we explained in the introduction, the fourth Painlevé equation (0.5) is associated with a Lax pair consisting of (SL_{IV}) and (D_{IV}), and a characteristic degeneracy of the Stokes geometry of (SL_{IV}) is observed when the parameter t lies in the Stokes curve of (0.5). (Cf figure 1.) Now, what if we consider another Lax pair (2.3) and (2.4) instead of the pair (SL_{IV}) and (D_{IV})? The computer-assisted study [13] has revealed a very intriguing fact A stated in the introduction. We also noted in the introduction that the introduction of virtual turning points brings back the harmony, as is stated in fact B. We now discuss how figures 3(i) and 3(ii) are bridged by the result in section 1, i.e., the relevance of a virtual turning point and the bifurcation phenomenon of a Stokes curve.

First of all, we note, by comparing figures 2(i) and 2(ii), the simple ordinary turning point s_2 should cross the Stokes curve connecting the turning points d and s_1 as t moves from t_1

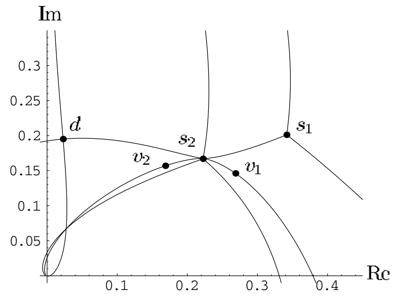

Figure 10. Configuration of Stokes curves of (2.3) when s_2 meets another Stokes curve.

to t_0 at, say $t = t_2$; at $t = t_2$ we find figure 10. Since s_2 is a simple ordinary turning point, bifurcation of Stokes curves is observed at $t = t_2$. The result in section 1 tells us that the bifurcation phenomenon is a counterpart of the addition of Stokes curves emanating from virtual turning points. In our situation, we can immediately find two virtual turning points v_1 and v_2; v_1 is related to the crossing of a Stokes curve emanating from s_1 and that from s_2, and v_2 is related to the crossing of a Stokes curve emanating from d and that from s_2, as the reader sees in figure 3(ii). In parenthesis, an interesting fact worth mentioning is that v_1 and v_2 are connected by a Stokes curve at $t = t_0$; this is not by accident. See appendix B for the analytic proof. Now, at $t = t_2$ the virtual turning point v_1 is connected both with the double turning point d and with the virtual turning point v_2 thanks to the bifurcation of the Stokes curve emanating from v_1, and similarly the virtual turning point v_2 is connected both with the simple turning point s_1 and with the virtual turning point v_1. When t moves further to reach $t = t_1$, the interchange of relative location of the Stokes curve emanating from v_1 and that from s_1 on the left of their crossing point switches the target of the Stokes curve (emanating from v_1) from v_2 to the double turning point d. In parallel with this, the virtual turning point v_2 is connected with the simple turning point s_1 by a Stokes curve at $t = t_1$. Thus we obtain figure 3(i); making a clear contrast with figure 2(i), the turning point d (resp., s_1) is connected with the turning point v_1 (resp., v_2). In conclusion, we find that the exact WKB analysis is applicable to the 3×3 system (2.3) despite the fact that we cannot manipulate it with ordinary WKB analysis.

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Appendix A. Coincidence of two Stokes curves

In this appendix we prove that the configuration observed in figure 8 in section 1, i.e., one new Stokes curve passing through two crossing points of Stokes curves simultaneously, is a phenomenon that is observed generically in our context.

In what follows (after placing a cut in an appropriate manner to define the characteristic roots $\zeta_2(x)$ and $\zeta_3(x)$ as single-valued analytic functions), we discuss the problem in the situation where a Stokes curve of type $1 > 2$ crosses two Stokes curves emanating from a simple ordinary turning point $x = s$ with a sign $(2, 3)$. The situation is visualized by figure 11, where c_j $(j = 1, 2)$ denote crossing points of Stokes curves, a solid line designates a Stokes

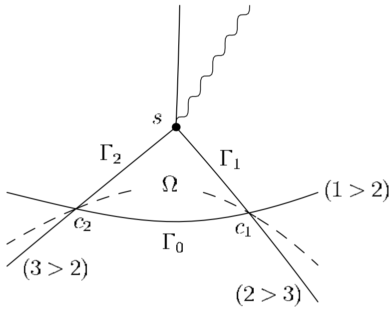

Figure 11. The situation where a Stokes curve crosses two Stokes curves emanating from a simple turning point.

curve and a wiggly line designates a cut. We also let Γ_0, Γ_1 and Γ_2 denote the closed portions of the Stokes curves $[c_1, c_2]$, $[s, c_1]$ and $[s, c_2]$, respectively. Here we assume that another Stokes curve, designated by a dotted line in figure 11, passing through c_j $(j = 1, 2)$ runs as is shown in figure 11, that is (one side of) it goes inside the triangle-like domain Ω surrounded by Γ_k. Our problem is then to find a generic condition which guarantees that these two Stokes curves passing through c_j coincide.

To describe such a generic condition, let us introduce the following terminology: when

$$\mathrm{Im}(\overline{(\zeta_1(x) - \zeta_3(x))}(\zeta_2(x) - \zeta_3(x))) \neq 0 \tag{A.1}$$

holds at a point x, we say that 'the transversality condition is satisfied at x'. Note that (A.1) is equivalent to

$$\mathrm{Im}(\overline{(\zeta_1(x) - \zeta_2(x))}(\zeta_3(x) - \zeta_2(x))) \neq 0 \tag{A.2}$$

and also to

$$\mathrm{Im}(\overline{(\zeta_2(x) - \zeta_1(x))}(\zeta_3(x) - \zeta_1(x))) \neq 0, \tag{A.3}$$

since

$$\overline{(\zeta_1 - \zeta_3)}(\zeta_2 - \zeta_3) + \overline{(\zeta_1 - \zeta_2)}(\zeta_3 - \zeta_2) = \overline{(\zeta_2 - \zeta_3)}(\zeta_2 - \zeta_3) \in \mathbb{R} \tag{A.4}$$

etc hold. The transversality condition means that any two of the three direction fields $\mathrm{Im}(\zeta_1(x) - \zeta_2(x))\,\mathrm{d}x = 0$, $\mathrm{Im}(\zeta_2(x) - \zeta_3(x))\,\mathrm{d}x = 0$ and $\mathrm{Im}(\zeta_3(x) - \zeta_1(x))\,\mathrm{d}x = 0$ are transversal to each other at a point x in question.

We now present a sufficient condition for the two Stokes curves passing through c_j to coincide. We first consider the case where the operator P in question contains no auxiliary parameter other than η.

Proposition A.1. *In the situation described in figure 11 we assume that, except at $x = s$, neither ordinary turning points nor singular points exist in a neighbourhood of the triangle-like closed domain Ω surrounded by Γ_k. We further assume that at some point $x = x_0 \in \Gamma_0\backslash\{c_1, c_2\}$*

$$\mathrm{Im}(\overline{(\zeta_1(x_0) - \zeta_2(x_0))}(\zeta_1(x_0) - \zeta_3(x_0))) = 0 \tag{A.5}$$

holds, while except at $x = x_0$ and $x = s$ the transversality condition is satisfied at every point of Γ_k $(k = 0, 1, 2)$. Then two Stokes curves, designated by dotted lines in figure 11, passing through c_j $(j = 1, 2)$ coincide.

Proof. Let $F_{jk}(x; x')$ and $f_{jk}(x)$ respectively denote $\int_{x'}^{x}(\zeta_j(z) - \zeta_k(z))\,\mathrm{d}z$ and its derivative $\zeta_j(x) - \zeta_k(x)$. To prove proposition A.1, we mainly consider $\mathrm{Im}F_{13}(x; c_1)$ in what follows.

First we note that $\mathrm{Im}F_{13}(c_2; c_1) = 0$ holds. As a matter of fact, since c_1 and c_2 (resp., s and c_1, s and c_2) are connected by a Stokes curve of type $1 > 2$ (resp., type $2 > 3$, type $3 > 2$), we have

$$\mathrm{Im}F_{12}(c_2; c_1) = \mathrm{Im}F_{23}(c_1; s) = \mathrm{Im}F_{23}(c_2; s) = 0, \tag{A.6}$$

and hence

$$\begin{aligned}\mathrm{Im}F_{13}(c_2; c_1) &= \mathrm{Im}F_{12}(c_2; c_1) + \mathrm{Im}F_{23}(c_2; c_1)\\ &= \mathrm{Im}F_{12}(c_2; c_1) + \mathrm{Im}F_{23}(c_2; s) - \mathrm{Im}F_{23}(c_1; s)\\ &= 0.\end{aligned} \tag{A.7}$$

Now, if we write as $x = u + \mathrm{i}v$ and $f_{jk} = g_{jk} + \mathrm{i}h_{jk}$, the tangential differentiation along a curve $\mathrm{Im}F_{jk}(x; c) = \mathit{const}.$ (c: a fixed point) is given by $g_{jk}\frac{\partial}{\partial u} - h_{jk}\frac{\partial}{\partial v}$. Hence, using the Cauchy–Riemann equation and the transversality assumption, we find that the tangential derivative of $\mathrm{Im}F_{13}(x; c_1)$ along a Stokes curve Γ_1, i.e. $\mathrm{Im}F_{23}(x; s) = 0$, should be

$$\begin{aligned}\left(g_{23}\frac{\partial}{\partial u} - h_{23}\frac{\partial}{\partial v}\right)\mathrm{Im}F_{13}(x; c_1) &= \mathrm{Im}\left(\overline{f_{23}(x)}\frac{\partial}{\partial x}F_{13}(x; c_1)\right)\\ &= \mathrm{Im}(\overline{(\zeta_2(x) - \zeta_3(x))}(\zeta_1(x) - \zeta_3(x)))\\ &\neq 0\end{aligned} \tag{A.8}$$

in the interior of Γ_1. This implies that $\mathrm{Im}F_{13}(x; c_1)$ is a monotone function and never vanishes on $\Gamma_1\backslash\{c_1\}$. On the other hand, a similar computation and equality (A.7) tell us that $\mathrm{Im}F_{13}(x; c_1)$ has one maximum (or minimum) at $x = x_0$ and vanishes only at the endpoints $x = c_1$ and $x = c_2$ on Γ_0. These behaviors of $\mathrm{Im}F_{13}(x; c_1)$ on Γ_1 and Γ_0 entail that the Stokes curve, which is defined by $\mathrm{Im}F_{13}(x; c_1) = 0$, passing through c_1 and going inside Ω should get out of Ω through Γ_2. (Note that the Stokes curve in question cannot stay inside Ω perpetually by the assumption and it must get out of Ω through the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$.)

By a similar reasoning we also find that the Stokes curve passing through c_2 and going inside Ω should get out of Ω through Γ_1. Hence, if the Stokes curve passing through c_1 (resp., c_2) is assumed to get out of Ω through $\Gamma_2\backslash\{c_2\}$ (resp., $\Gamma_1\backslash\{c_1\}$), then the two Stokes curves are forced to intersect inside Ω, which is a contradiction since these two curves are solution curves of the same direction field $\mathrm{Im}(\zeta_1(x) - \zeta_3(x))\,\mathrm{d}x = 0$. Thus the Stokes curve passing through c_1 must get out of Ω through c_2. This completes the proof of proposition A.1. □

Next we discuss the situation which we encountered in figure 8, that is, the case where the operator P contains an auxiliary parameter ('deformation parameter') t. (Consequently the turning point $x = s$ etc also depend on t; to manifest this dependence on t, we use the notation $x = s(t)$ in what follows.) As in section 1, let us assume that at $t = t_1$ a Stokes curve of type $1 > 2$ hits a simple ordinary turning point $s(t_1)$ and that at $t = t_3$, a point sufficiently close to $t = t_1$, the configuration of Stokes curves is that given in figure 11. Then in this situation we can prove the following

Proposition A.2. *Assume that at $t = t_1$ a Stokes curve of type $1 > 2$ hits a simple ordinary turning point $s(t_1)$ and that the Stokes curve of type $1 > 2$ is not tangential at $x = s(t_1)$ to any of the three Stokes curves emanating from $s(t_1)$. Then at $t = t_3$, which is sufficiently close to $t = t_1$ and where the configuration of Stokes curves is that given in figure 11, two Stokes curves, designated by dotted lines in figure 11, passing through $c_j = c_j(t_3)(j = 1, 2)$ coincide.*

Proof. Let $N = N(t)$ denote the set of non-transversal points, i.e.,

$$N = \{x; \mathrm{Im}(\overline{(\zeta_1(x) - \zeta_3(x))}(\zeta_2(x) - \zeta_3(x))) = 0\}. \tag{A.9}$$

To prove proposition A.2, we first investigate the local configuration of N near $x = s(t)$ at $t = t_1$.

Since $x = s(t_1)$ is a simple turning point where $\zeta_2(x)$ and $\zeta_3(x)$ coalesce and $\zeta_2(x)$ (together with $\zeta_3(x)$) has a square-root-type singularity there, we readily find that N is a curved ray emanating from $s(t_1)$ in a neighbourhood of $s(t_1)$. Furthermore, if we write the Puiseux expansions of the characteristic roots $\zeta_j(x)$ at $x = s(t_1)$ as

$$\zeta_1(x) = a_0 + a_2(x - s(t_1)) + \cdots, \tag{A.10}$$

$$\zeta_2(x) = b_0 + b_1(x - s(t_1))^{1/2} + b_2(x - s(t_1)) + \cdots, \tag{A.11}$$

$$\zeta_3(x) = b_0 - b_1(x - s(t_1))^{1/2} + b_2(x - s(t_1)) - \cdots, \tag{A.12}$$

the tangent vector $s(t_1) + r\,\mathrm{e}^{\mathrm{i}\theta_N}$ $(0 < r \ll 1)$ of N at $x = s(t_1)$ is determined by the following equality:

$$\mathrm{Im}(\overline{(a_0 - b_0)}b_1\,\mathrm{e}^{\mathrm{i}\theta_N/2}) = 0. \tag{A.13}$$

That is, letting $r_0\,\mathrm{e}^{\mathrm{i}\theta_0}$ and $r_1\,\mathrm{e}^{\mathrm{i}\theta_1}$ respectively denote $a_0 - b_0$ and b_1 (both of which are non-zero complex numbers by the assumption), we obtain

$$\theta_N \equiv 2(\theta_0 - \theta_1) \pmod{2\pi\mathbb{Z}}. \tag{A.14}$$

On the other hand, under these notations, at $x = s(t_1)$ the tangent vector $s(t_1) + r\,\mathrm{e}^{\mathrm{i}\theta_{12}}$ $(0 < r \ll 1)$ of the Stokes curve of type $1 > 2$ hitting $s(t_1)$ and the tangent vector $s(t_1) + r\,\mathrm{e}^{\mathrm{i}\theta_{23}}$ $(0 < r \ll 1)$ of the Stokes curves (of type (2, 3)) emanating from $s(t_1)$ are respectively given by

$$\theta_{12} \equiv -\theta_0, -\theta_0 + \pi \pmod{2\pi\mathbb{Z}}, \tag{A.15}$$

$$\theta_{23} \equiv -\frac{2}{3}\theta_1, -\frac{2}{3}\theta_1 + \frac{2\pi}{3}, -\frac{2}{3}\theta_1 + \frac{4\pi}{3} \pmod{2\pi\mathbb{Z}}. \tag{A.16}$$

Now let us investigate the relative location of the tangent vector $s(t_1) + r\,\mathrm{e}^{\mathrm{i}\theta_N}$ of N and the tangent vectors $s(t_1) + r\,\mathrm{e}^{\mathrm{i}\theta_{12}}$ and $s(t_1) + r\,\mathrm{e}^{\mathrm{i}\theta_{23}}$ of the Stokes curves in question. If we introduce a new angular variable $\tilde{\theta} = \theta + \frac{2}{3}\theta_1$, these tangent vectors are respectively given by

$$\tilde{\theta}_N \equiv -2\hat{\theta} \pmod{2\pi\mathbb{Z}}, \tag{A.17}$$

$$\tilde{\theta}_{12} \equiv \hat{\theta}, \hat{\theta} + \pi \pmod{2\pi\mathbb{Z}}, \tag{A.18}$$

$$\tilde{\theta}_{23} \equiv 0, \frac{2\pi}{3}, \frac{4\pi}{3} \pmod{2\pi\mathbb{Z}}, \tag{A.19}$$

where $\hat{\theta} = \frac{2}{3}\theta_1 - \theta_0$, in terms of this new angular variable $\tilde{\theta}$. Here, without loss of generality, we may assume $0 \leqslant \hat{\theta} < \pi$. Furthermore $\hat{\theta} \neq \frac{k}{3}\pi$ $(k \in \mathbb{Z})$ holds by the assumption. We can then divide the problem into the following three cases:

$$0 < \hat{\theta} < \frac{\pi}{3}, \quad \frac{\pi}{3} < \hat{\theta} < \frac{2\pi}{3}, \quad \frac{2\pi}{3} < \hat{\theta} < \pi, \tag{A.20}$$

and, looking into the relative location of $\tilde{\theta}_N$, $\tilde{\theta}_{12}$ and $\tilde{\theta}_{23}$ in each case, we can confirm that the local configuration of N and the Stokes curves in question near $x = s(t_1)$ is that given in figure 12 in all cases. (In figure 12 the non-transversal set N (resp., a Stokes curve) is designated by a thick line (resp., thin line).)

Hence, if $t = t_3$ is sufficiently close to $t = t_1$ and the configuration of Stokes curves at $t = t_3$ is that given in figure 11, all the assumptions of proposition A.1 are satisfied. Thus, applying proposition A.1, we obtain the desired coincidence of the two Stokes curves at $t = t_3$, which completes the proof of proposition A.2. □

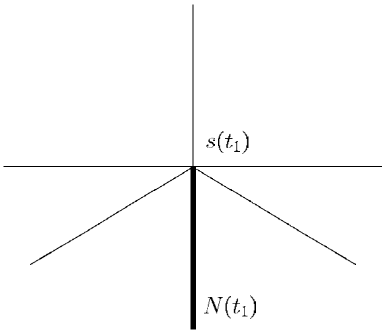

Figure 12. Local (relative) configuration of *N* and the Stokes curves.

Appendix B. Expression of the phase function $\phi(t)$ with the help of virtual turning points

As mentioned in the introduction, the phase function $\phi(t)$ of the instanton-type expansion of the Painlevé transcendent [9] is given by an integral (0.9) whose endpoints are a double ordinary turning point d and a simple ordinary turning point s of (SL_{IV}) when the Lax pair attached to (P_{IV}) is given by (SL_{IV}) and (D_{IV}). When equations (2.3) and (2.4) are used as another Lax pair underlying (P_{IV}), we have found in section 2 that the ordinary turning points corresponding to d and s in the case of (SL_{IV}) are superseded by virtual turning points when the parameter t lies in some portion of a Stokes curve for (P_{IV}) in that the expected degeneracy of the Stokes geometry of (2.3), which is drawn in x-space, is realized by connecting an ordinary turning point and a virtual turning point of (2.3). Furthermore, the degeneracy is realized by two pairs of an ordinary turning point and a virtual turning point at once. Thus an analytically important question naturally arises: how are these virtual turning points related to the phase function? The purpose of appendix B is to answer this question. The final results are summarized in theorem B.1 at the end of appendix B.

Before stating the theorem let us recall some local properties of virtual turning points. All the computations in [13] and [5] for locating virtual turning points are done on the local properties described below. See [5] for the background of this analysis. To begin with, let us concentrate our attention on a crossing point C of a Stokes curve γ_1 of type $(1, 2)$ that emanates from a turning point τ_1 and another Stokes curve γ_2 of type $(2, 3)$ that emanates from a turning point τ_2. Here τ_1 and τ_2 may be either virtual or ordinary. (When τ_1 or τ_2 is a simple ordinary turning point, we need to introduce an appropriate cut to fix the branch of the characteristic roots.) Then it follows from the definition that the bicharacteristic curves passing through $(x, y) = (\tau_1, 0)$ are given either by

$$y = -\int_{\tau_1}^{x} \xi_1 \mathrm{d}x \tag{B.1}$$

or

$$y = -\int_{\tau_1}^{x} \xi_2 \, \mathrm{d}x. \tag{B.2}$$

Here and in what follows we normalize η to be 1 in equation (1.5).

Let y_* denote

$$-\int_{\tau_1}^{x_*} \xi_1 \, \mathrm{d}x \tag{B.3}$$

and consider the following succession of journeys along bicharacteristic curves:

(i) Start from (x_*, y_*) and reach $(\tau_1, 0)$ along the curve given by (B.1).

(ii) Continue the journey from $(\tau_1, 0)$ to reach $p = \left(\tau_2, -\int_{\tau_1}^{\tau_2} \xi_2 \mathrm{d}x\right)$ along the curve defined by (B.2), i.e.,

$$y = -\int_{\tau_1}^{x} \xi_2 \,\mathrm{d}x = -\int_{\tau_2}^{x} \xi_2 \,\mathrm{d}x - \int_{\tau_1}^{\tau_2} \xi_2 \,\mathrm{d}x. \tag{B.4}$$

(iii) Switch our journey to another bicharacteristic curve merging with the curve given by (B.4) at p, that is, continue our journey along the bicharacteristic curve given by

$$y = -\int_{\tau_2}^{x} \xi_3 \,\mathrm{d}x - \int_{\tau_1}^{\tau_2} \xi_2 \,\mathrm{d}x. \tag{B.5}$$

If we come back to our starting point (x_*, y_*) after these journeys, that is, if

$$\int_{\tau_1}^{x_*} \xi_1 \,\mathrm{d}x = \int_{\tau_1}^{\tau_2} \xi_2 \,\mathrm{d}x + \int_{\tau_2}^{x_*} \xi_3 \,\mathrm{d}x \tag{B.6}$$

holds, then it follows from the definition that x_* is a virtual turning point. We next consider how this virtual turning point x_* is related to the crossing point C of Stokes curves γ_1 and γ_2. For this purpose let us consider the following integral I:

$$I = \int_{\tau_1}^{x_*} (\xi_1 - \xi_2) \,\mathrm{d}x. \tag{B.7}$$

It then follows from (B.6) that

$$\begin{aligned} I &= \int_{\tau_1}^{\tau_2} \xi_2 \,\mathrm{d}x + \int_{\tau_2}^{x_*} \xi_3 \,\mathrm{d}x - \int_{\tau_1}^{x*} \xi_2 \,\mathrm{d}x \\ &= \int_{\tau_2}^{x_*} (\xi_3 - \xi_2) \,\mathrm{d}x. \end{aligned} \tag{B.8}$$

Combining (B.7) and (B.8) we obtain

$$\int_{\tau_1}^{C} (\xi_1 - \xi_2) \,\mathrm{d}x + \int_{C}^{x_*} (\xi_1 - \xi_2) \,\mathrm{d}x = \int_{\tau_2}^{C} (\xi_3 - \xi_2) \,\mathrm{d}x + \int_{C}^{x_*} (\xi_3 - \xi_2) \,\mathrm{d}x. \tag{B.9}$$

Hence we find

$$\int_{C}^{x_*} (\xi_1 - \xi_3) \,\mathrm{d}x = \int_{\tau_1}^{C} (\xi_2 - \xi_1) \,\mathrm{d}x + \int_{\tau_2}^{C} (\xi_3 - \xi_2) \,\mathrm{d}x. \tag{B.10}$$

Since C is a crossing point of the Stokes curves γ_1 and γ_2, the imaginary part of the right-hand side of (B.10) vanishes, and hence so does the imaginary part of the left-hand side. This implies that C lies in the Stokes curve of type $(1, 3)$ emanating from the virtual turning point x_*. (Strictly speaking, we have to confirm, in addition, that C and x_* belong to the same connected component of the curve defined by $\operatorname{Im} \int_{x_*}^{x} (\xi_1 - \xi_3) \,\mathrm{d}x = 0$. This topological condition, however, is easy to confirm in the computer-assisted study.) This geometric study can be converted to another characterization of the virtual turning point in terms of the crossing point C, which is the most convenient one in our analysis. For this purpose we consider a curve γ defined by

$$\operatorname{Im} \int_{C}^{x} (\xi_1 - \xi_3) \,\mathrm{d}x = 0. \tag{B.11}$$

Let $\rho(x)$ denote

$$\operatorname{Re} \int_{C}^{x} (\xi_1 - \xi_3) \,\mathrm{d}x. \tag{B.12}$$

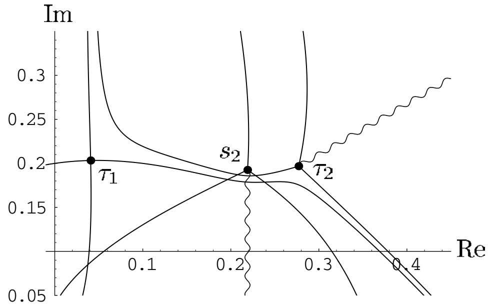

Figure 13. A small perturbation of figure 2(ii).

Then $\rho(x)$ is, as the real part of a holomorphic function, monotonically decreasing or increasing along the real one-dimensional curve γ. Hence we can normally (i.e., barring the exceptional situation where ρ is bounded on γ) locate a point x_0 in γ where $\rho(x)$ attains the real number

$$\int_{\tau_1}^{C} (\xi_2 - \xi_1)\,\mathrm{d}x + \int_{\tau_2}^{C} (\xi_3 - \xi_2)\,\mathrm{d}x. \tag{B.13}$$

Then x_0 satisfies

$$\int_{C}^{x_0} (\xi_1 - \xi_3)\,\mathrm{d}x = \int_{\tau_1}^{C} (\xi_2 - \xi_1)\,\mathrm{d}x + \int_{\tau_2}^{C} (\xi_3 - \xi_2)\,\mathrm{d}x. \tag{B.14}$$

Comparing (B.10) and (B.14) we conclude that x_0 and x_* should coincide. Otherwise stated, we can find a virtual turning point x_* that satisfies the relation (B.15) below in our context:

$$\int_{C}^{x_*} (\xi_1 - \xi_3)\,\mathrm{d}x = \int_{C}^{\tau_1} (\xi_1 - \xi_2)\,\mathrm{d}x + \int_{C}^{\tau_2} (\xi_2 - \xi_3)\,\mathrm{d}x. \tag{B.15}$$

Since all the virtual turning points we use in our subsequent discussion, namely the virtual turning points v_1 and v_2 introduced in section 2 and another virtual turning point v_3 to be used below, are located by this method, we regard (B.15) as the defining equation of a virtual turning point in what follows. Since the integrands of the integrals considered below may be assumed to be non-singular except for an endpoint (i.e., a simple ordinary turning point), we slightly perturb the parameter t so that the degeneracy observed in figure 2(ii) becomes as is described in figure 13. Here and in what follows τ_1 is a double turning point and τ_2 is a simple turning point. We use a wiggly line to designate a cut attached to a simple turning point. We then add virtual turning points v_1, v_2 and v_3 to this dispersed figure to find figure 14; v_1 and v_2 are respectively slight perturbations of v_1 and v_2 in figure 3(ii) and the virtual turning point v_3 was omitted in figure 3(ii) for simplicity. In what follows the dotted line is used to specify the portion of a Stokes curve which is inert concerning Stokes phenomena, although in understanding the computation below the reader may ignore the difference between the dotted line and the solid line. We also note that the virtual turning point v_3 plays an important role in relating several integrals discussed below.

Using (B.15) we find that the virtual turning points v_j ($j = 1, 2, 3$) satisfy the following relations, where C_j ($j = 1, 2, 3$) denotes the crossing point of Stokes curves designated in figure 15.

$$\int_{v_1}^{C_1} (\xi_3 - \xi_1)\,\mathrm{d}x = \int_{\tau_2}^{C_1} (\xi_2 - \xi_1)\,\mathrm{d}x + \int_{s_2}^{C_1} (\xi_3 - \xi_2)\,\mathrm{d}x, \tag{B.16}$$

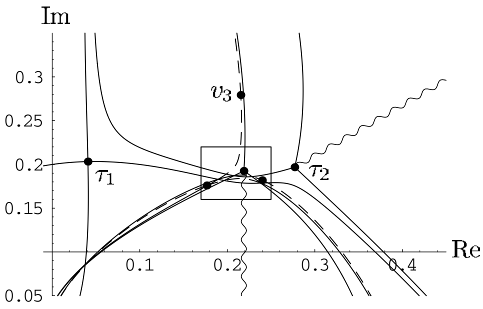

Figure 14. Figure 13 with virtual turning points added.

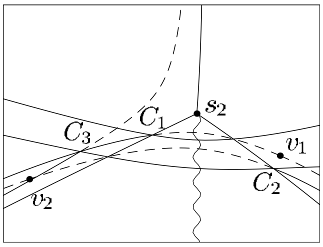

Figure 15. Magnification of the portion enclosed by a square in figure 14.

$$\int_{v_2}^{C_2} (\xi_1 - \xi_3)\,\mathrm{d}x = \int_{\tau_1}^{C_2} (\xi_1 - \xi_2)\,\mathrm{d}x + \int_{s_2}^{C_2} (\xi_2 - \xi_3)\,\mathrm{d}x, \tag{B.17}$$

$$\int_{v_3}^{C_3} (\xi_3 - \xi_2)\,\mathrm{d}x = \int_{\tau_1}^{C_3} (\xi_1 - \xi_2)\,\mathrm{d}x + \int_{v_1}^{C_3} (\xi_3 - \xi_1)\,\mathrm{d}x. \tag{B.18}$$

Adding (B.17) and (B.16) we obtain

$$\int_{v_1}^{v_2} (\xi_3 - \xi_1)\,\mathrm{d}x - \int_{C_1}^{C_2} (\xi_3 - \xi_1)\,\mathrm{d}x = \int_{\tau_2}^{\tau_1} (\xi_2 - \xi_1)\,\mathrm{d}x - \int_{C_1}^{C_2} (\xi_2 - \xi_1)\,\mathrm{d}x + \int_{C_1}^{C_2} (\xi_2 - \xi_3)\,\mathrm{d}x, \tag{B.19}$$

and hence

$$\int_{v_1}^{v_2} (\xi_3 - \xi_1)\,\mathrm{d}x = \int_{\tau_2}^{\tau_1} (\xi_2 - \xi_1)\,\mathrm{d}x. \tag{B.20}$$

Thus we have seen the integral corresponding to (0.9) is equal to an integral whose endpoints are virtual turning points. Furthermore this integral is also equal to

$$\int_{s_2}^{v_3} (\xi_2 - \xi_3)\,\mathrm{d}x. \tag{B.21}$$

In fact, (B.16) entails

$$\int_{\tau_2}^{v_1} \xi_1\,\mathrm{d}x = \int_{\tau_2}^{s_2} \xi_2\,\mathrm{d}x + \int_{s_2}^{v_1} \xi_3\,\mathrm{d}x, \tag{B.22}$$

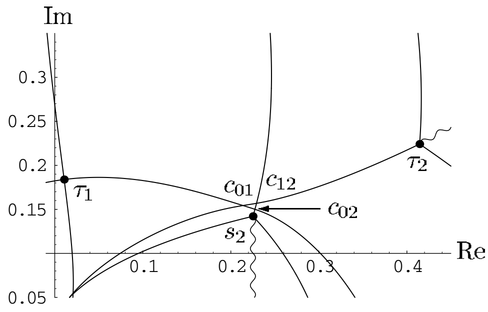

Figure 16. A small perturbation of figure 2(i).

and (B.18) entails

$$\int_{\tau_1}^{v_3} \xi_2 \, dx = \int_{\tau_1}^{v_1} \xi_1 \, dx + \int_{v_1}^{s_2} \xi_3 \, dx + \int_{s_2}^{v_3} \xi_3 \, dx. \tag{B.23}$$

Then, by combining (B.22) and (B.23), we find

$$\int_{\tau_1}^{v_1} \xi_1 \, dx + \int_{v_1}^{\tau_2} \xi_1 \, dx - \int_{\tau_1}^{v_3} \xi_2 \, dx + \int_{\tau_2}^{s_2} \xi_2 \, dx + \int_{s_2}^{v_3} \xi_3 \, dx = 0, \tag{B.24}$$

i.e.,

$$\int_{\tau_1}^{\tau_2} (\xi_1 - \xi_2) \, dx = \int_{s_2}^{v_3} (\xi_2 - \xi_3) \, dx. \tag{B.25}$$

Next we disperse figure 2(i) by an appropriate small perturbation of t; then we obtain figure 16. In figure 16 we see a tiny triangle formed by Stokes curves that is located close to s_2: let c_{01}, c_{02} and c_{12} respectively designate the left, right-down and right-up vertex of the triangle. Now let us consider the following sum $J(x)$ of integrals:

$$J(x) = \int_{\tau_1}^{x} (\xi_1 - \xi_2) \, dx + \int_{\tau_2}^{x} (\xi_3 - \xi_1) \, dx + \int_{s_2}^{x} (\xi_2 - \xi_3) \, dx. \tag{B.26}$$

Since the derivative of $J(x)$ vanishes identically, $J(x)$ is actually a constant depending on the parameter t. Furthermore, as we see below, $J(c_{01})$ is equal to the integral of the form (B.21), and we can also confirm

$$J(c_{12}) = \int_{\tau_1}^{v_1} (\xi_1 - \xi_2) \, dx \tag{B.27}$$

and

$$J(c_{02}) = \int_{\tau_2}^{v_2} (\xi_3 - \xi_1) \, dx \tag{B.28}$$

if virtual turning points v_1, v_2 and v_3 added to figure 16. These virtual turning points are deformations of corresponding ones in figure 14. In fact, by using (B.15), we find

$$\int_{v_3}^{c_{01}} (\xi_3 - \xi_2) \, dx = \int_{\tau_1}^{c_{01}} (\xi_1 - \xi_2) \, dx + \int_{\tau_2}^{c_{01}} (\xi_3 - \xi_1) \, dx \tag{B.29}$$

etc. Hence we obtain

$$J(c_{01}) = \int_{v_3}^{c_{01}} (\xi_3 - \xi_2) \, dx + \int_{s_2}^{c_{01}} (\xi_2 - \xi_3) \, dx = \int_{s_2}^{v_3} (\xi_2 - \xi_3) \, dx. \tag{B.30}$$

Relations (B.27) and (B.28) can be confirmed by the same reasoning. Thus we obtain the following

Theorem B.1. *(i) On a neighbourhood of the point $t = t_0$, we find*

$$\int_{\tau_1(t)}^{\tau_2(t)} (\xi_1(x,t) - \xi_2(x,t))\,\mathrm{d}x = \int_{v_1(t)}^{v_2(t)} (\xi_3(x,t) - \xi_1(x,t))\,\mathrm{d}x$$
$$= \int_{s_2(t)}^{v_3(t)} (\xi_2(x,t) - \xi_3(x,t))\,\mathrm{d}x. \tag{B.31}$$

(ii) On a neighbourhood of the point $t = t_1$, we find

$$\int_{s_2(t)}^{v_3(t)} (\xi_2(x,t) - \xi_3(x,t))\,\mathrm{d}t = \int_{\tau_1(t)}^{v_1(t)} (\xi_1(x,t) - \xi_2(x,t))\,\mathrm{d}t$$
$$= \int_{\tau_2(t)}^{v_2(t)} (\xi_3(x,t) - \xi_1(x,t))\,\mathrm{d}t. \tag{B.32}$$

Remark B.1. (i) The first integral of (B.31) is known [9, 11] to coincide with the phase function $\phi(t)$ in the instanton expansion of solutions near a turning point of the Painlevé equation, or more generally, its higher order version $(NY)_l$, the Noumi–Yamada system [12]. This representation of the phase function should be, however, replaced by one of the integrals given in (B.32) near the point $t = t_1$. An important fact to be noted here is that one of the endpoints in each integral is a virtual turning point; virtual turning points are indispensable in the exact WKB analysis of the Noumi–Yamada system.
(ii) It follows from (B.31) that

$$\mathrm{Im} \int_{v_1(t)}^{v_2(t)} (\xi_3(x,t) - \xi_1(x,t))\,\mathrm{d}x = 0 \tag{B.33}$$

when

$$\mathrm{Im} \int_{\tau_1(t)}^{\tau_2(t)} (\xi_1(x,t) - \xi_2(x,t))\,\mathrm{d}x = 0. \tag{B.34}$$

This explains why the virtual turning points $v_1(t)$ and $v_2(t)$ are connected by a Stokes segment in figure 3(ii).

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